Minimal Model Program

Learning Seminar.

Week 29.

Contents:

- Cox rings.
- Mori dream spaces.

Mori dream spaces:

In $\mathbb{P}^n_{\mathbb{K}}$, every Weil divisor $W \sim dH$, then

$$H^0(\mathbb{P}^n, W) \simeq H^0(\mathbb{P}^n, dH) \simeq \mathbb{K}[x_0^d, x_0^{d-1}x_1, \ldots, x_n^d] \quad \text{free } \mathbb{K}\text{-module}$$

Thus, we have that

$\mathbb{K}$-ring.

$$\bigoplus_{[W] \in Cl(\mathbb{P}^n)} H^0(\mathbb{P}^n, W) \simeq \mathbb{K}[x_0, \ldots, x_n]$$

Any hypersurface in $\mathbb{P}^n$ corresponds to an homogeneous poly of The homogeneous coordinates of $\mathbb{P}^n$.

$$\operatorname{Spec}(\mathbb{K}[x_0, \ldots, x_n] - \{(0, \ldots, 0)\}) / \mathbb{G}_m = \mathbb{P}^n.$$

Free + f.g class group:

X normal proj variety with $Cl(X) \simeq \mathbb{Z}^r$ free and f.g

$K \leq WDiv(X)$, $K \longrightarrow Cl(X)$ is an isomorphism.

Then we define:

$$Cox(X) := \bigoplus_{W \in K} H^0(X, W)$$

We multiply sections in $\mathbb{K}(X)$.

Example: $Cox(\mathbb{P}^1 \times \mathbb{P}^1) \simeq \mathbb{K}[x_0, x_1, y_0, y_1]$.

graded by the bi-degree on x and y.

Cox ring: Assume $Cl(X)$ is f.g.

X normal + proj

$K \leq WDiv(X)$ surjecting onto $Cl(X)$

K_0 the kernel of $K \xrightarrow{c} Cl(X)$.

$\chi : K_0 \longrightarrow \mathbb{K}(X)^*$ be a homomorphism yielding

$$\operatorname{div}(\chi(E)) = E$$

for all $E \in K_0$.

$$\mathcal{I} = \langle 1 - \chi(E) \mid E \in K_0 \rangle.$$

$D \in K$, we define $S_D := \mathcal{O}_X(D)$.

The Cox sheaf of X is defined to be $\mathcal{R} = S/\mathcal{I}$

$$\mathcal{R} = \bigoplus_{D \in Cl(X)} \mathcal{R}_D \qquad \mathcal{R}_D := \pi\Big(\bigoplus_{D' \in c^{-1}(\{D\})} S_{D'} \Big).$$

The Cox ring $Cox(X) := \Gamma(X, \mathcal{R})$

which admits a $Cl(X)$-grading

Index one cover:

D on X, $\underbrace{mD \sim 0}$.

$$Y = \operatorname{Spec}\left(\mathcal{O}_X \oplus \mathcal{O}_X(D) \oplus \dots \oplus \mathcal{O}_X((m-1)D)\right)$$

$$\mathcal{O}_X(iD) \otimes \mathcal{O}_X(jD) = \mathcal{O}_X((i+j)D) = \mathcal{O}_X((i+j) \bmod m\, D).$$

$$Y \longrightarrow X$$

This is an example of a "Cox ring".

Main questions:

- Is $Cl(X)$ f.g? yes
- Is $Cox(X)$ f.g?

Remark: The set of isomorphisms of $Cox(X)$ depends on $\chi : K_0 \longrightarrow \mathbb{K}(X)^\times$. The set of isomorphism classes is in bijection to

$$\operatorname{Ext}^1(Cl(X), \mathcal{O}(X)^*).$$

Mori dream space:

MMP "=" Mori Theory.

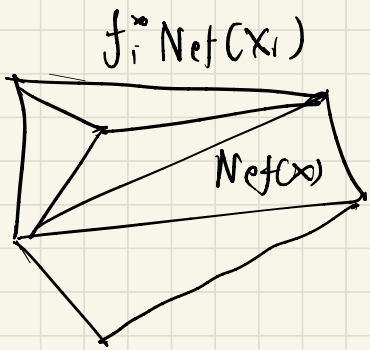
Def: A normal proj variety X is a MDS if

(1) X is $\mathbb{Q}$-factorial and $\mathrm{Pic}(X)_{\mathbb{Q}} = N^1(X)$; ✓✓

(2) $\mathrm{Nef}(X)$ is the affine hull of finitely many semiample line bundles

(3) There are finitely many small $\mathbb{Q}$-factorial models $f_i : X \dashrightarrow X_i$ such that each X_i satisfies (2) and

$$\mathrm{Mov}(X) = \bigcup_i f_i^* \mathrm{Nef}(X_i)$$



$= \mathrm{Mov}(X)$

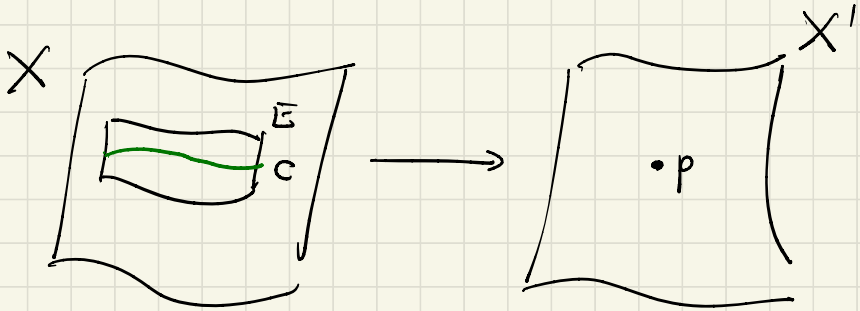
Proposition: Let X be a MDS. Then the following conditions hold:

(1) Mori's program can be carried out for any divisor on X:

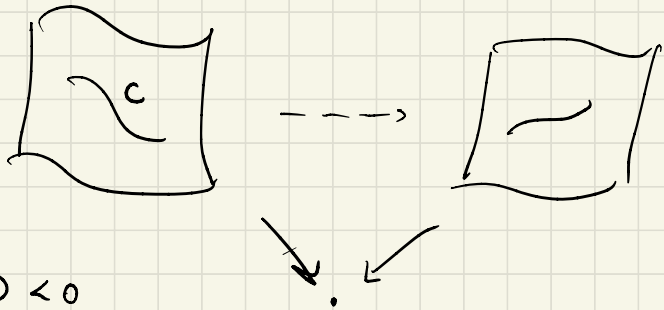
(2) D is a pseudo-eff on X, there exists a sequence

$$X \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_k$$

of D-divisorial contractions and D-flips, so that the strict transform of D on X_k is semiample



$$C.D < 0, \quad \rho(X/X') = 1$$



flipping curve intersects D negatively

$C.D < 0$

(b) D is not pseff on X. There exists a sequence

$$\begin{tikzcd} X \arrow[r, dashed] & X_1 \arrow[r, dashed] & X_2 \arrow[r, dashed] & \cdots \arrow[r, dashed] & X_k \arrow[d] \\ & & & & Z. \end{tikzcd}$$

such that each $X_i \dashrightarrow X_{i+1}$ is a D-flip, or a D-divisorial contraction. Moreover $X_k \longrightarrow Z$ is a D_k-Mori fiber space

D_k is the strict transform of D in X_k.

(2). There are finitely many birational contractions $g_i : X \dashrightarrow Y_i$ with Y_i a MDS s.t.:

$$\overline{NE}^1(X) = \bigcup_i g_i^*(\mathrm{Nef}(Y_i)) \times \mathrm{ex}(g_i)$$

In particular, $\overline{NE}^1(X)$ is rat polyhedral.

(3) The chambers $f_i^*(\mathrm{Nef}(X_i))$ together with their faces, give a fan structure of $\mathrm{Mov}(X)$.

These cones are in one-to-one correspondence with rational maps $g : X \dashrightarrow Y$ with Y normal + proj.

via $$[g : X \dashrightarrow Y] \longmapsto [g^*(\mathrm{Nef}(Y)) \subseteq \mathrm{Mov}(X)]$$

Example of a MDS:

$\mathbb{P}^3$, $X = Bl_{p,q}\,\mathbb{P}^3$. $\boxed{\rho_X = 3.}$

ℓ be the strict transform of the line through p & q.

Normal bundle of ℓ is $\mathcal{O}_{\mathbb{P}}(-1)^{\oplus 2}$; This curve can be contracted via a small contraction. $f: X \longrightarrow Y$.

$\hat{X}$ be the blow-up of ℓ. and $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ and its normal bundle $\mathcal{O}(-1,-1)$. We can contract E in the second direction to obtain a smooth blow-down:

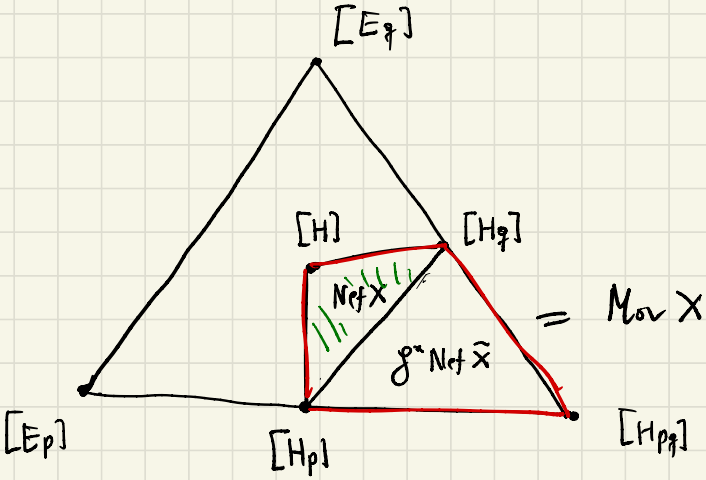
$$\begin{tikzcd}
 & \hat{X} \arrow[dl] \arrow[dr] \arrow[r, bend left, "\text{2nd contraction}"] & \\
\ell \subseteq X \arrow[rr, dashed, "g"] \arrow[dr, "f"'] & & \tilde{X} \supseteq \tilde{\ell} \arrow[dl] \\
 & Y &
\end{tikzcd}$$

Example of MDS:

$\tilde{X} \longrightarrow \mathbb{P}^1$ is a smooth morphism with fiber $\mathbb{P}^1 \times \mathbb{P}^1$.

(birational proj $\mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ from the line through p and q)

The morphism $\tilde{X} \longrightarrow \mathbb{P}^1$ factors in two different ways through a $\mathbb{P}^1$-bundle $\tilde{X} \longrightarrow \mathbb{F}_1$.



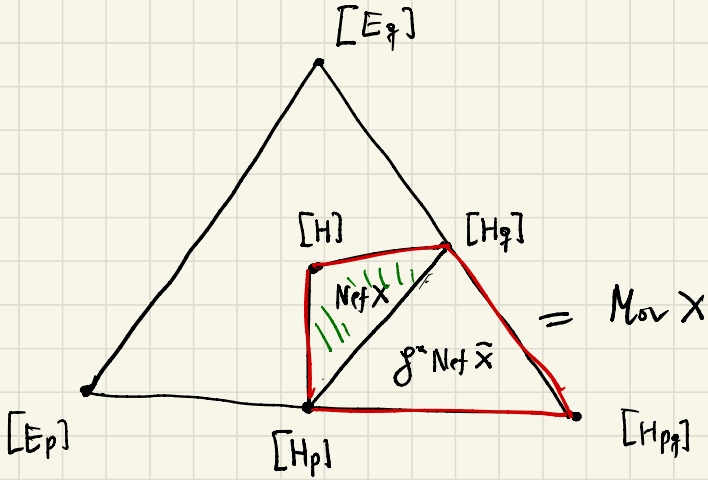
3-dim faces: $X \xrightarrow{id} X$, $X \dashrightarrow \tilde{X}$

2-dim faces:

corresponds to $X \longrightarrow Y$

corresponds to $X \longrightarrow Bl_q \mathbb{P}^2$ & $X \longrightarrow Bl_p \mathbb{P}^2$.

corresponds to $X \dashrightarrow \mathbb{F}_1$.



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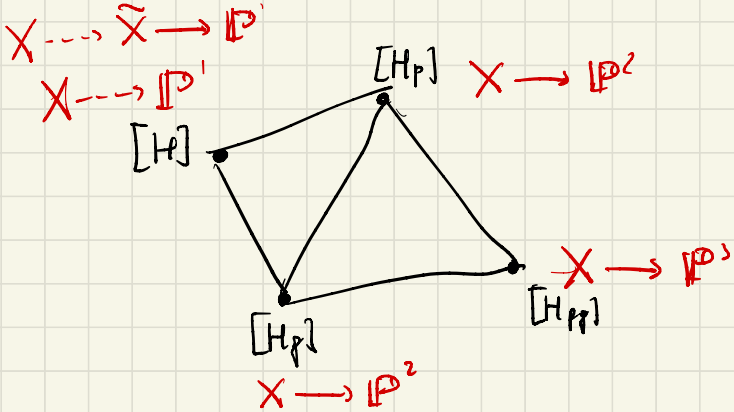

Mori dream space:

Thm (Hu & Keel 2000's): X is a MDS if and only if $\mathrm{Cox}(X)$ is finitely generated.

Good geometric properties of MDS:

- Any nef divisor is semiample.
- Any pseff divisor is effective
- The image of a MDS is also a MDS.

$\mathbb{G}_m^k \times A$
$\parallel$
$(\mathbb{K}^*)^k \times A$.

Abelian quasi-torsor: X normal variety.

$Y \longrightarrow X$ is an abelian quasi-torsor if there exists H a quasi-torus acting on Y satisfying the following conditions:

(i) $H_0 \leq H$, $Y' = Y/H_0 \longrightarrow X$ is quasi-étale,

(ii) $U_Y \subseteq Y$, $U_{Y'} \subseteq Y'$ big open sets so that $U_Y \longrightarrow U_{Y'}$ is étale locally a trivial H_0-bundle

(iii) $\mathcal{O}(Y)^H \simeq \mathcal{O}(X)$.

~~finite~~ cover
quasi-torus

finite covers $\rightsquigarrow$ universal cover.

abelian quasi-torsors $\rightsquigarrow$ spectrum of $\mathrm{Cox}(X)$.

Proposition: The spectrum of the Cox ring of X is an universal abelian quasi-torsor.

Corollary: $\mathbb{Q}$-points of X correspond to $\mathbb{Z}$-points of $\mathrm{Spec}(\mathrm{Cox}(X))$.

$$X, \quad \mathrm{Cox}(X), \quad \mathrm{Cox}(\mathrm{Cox}(X)), \ldots$$

height of $\mathbb{Q}$-points of X increases with the iteration.

characteristic quasi-torus

Corollary: $\mathbb{T} = \mathrm{Spec}(\mathbb{K}[\mathrm{Cl}(X)])$ of a MDS X

Then $X \simeq (\mathrm{Spec}(\mathrm{Cox}(X)) - V) /\!/ \mathbb{T}$.

closed of codim ≥ 2

Any Weil divisor on X corresponds to a homogeneous regular function on $\mathrm{Cox}(X)$.

Local Mori dream spaces:

$$\mathrm{Cox}(X) = \underbrace{\bigoplus_{D \in \mathrm{Cl}(X)}}\, H^0(X, D)$$

(E, p), $\mathrm{Cl}(U)$ is not f.g for any $\overset{\infty}{\overset{\cap}{U}} \subseteq E$

(X, x), you try to use the local group $\mathrm{Cl}(X_x)$.

$$\mathrm{Cox}(X; x) = \bigoplus_{D \in \mathrm{Cl}(X_x)} H^0(X_x, D)$$

Example: $\{(x, y, z, w) \mid x^2 + y^3 + z^3 w = 0\} = X$

$(0,0,0,0) = x$.

Sing curve $C = \{x = y = z = 0\}$, $c \in C$ general.

X étale around c is isomorphic to $\mathbb{C} \times E_6$ sing.

The Class group $\mathrm{Cl}(X_x)$ is trivial, $\mathrm{Cl}(X)$ is trivial.

However, $\pi_1(X^{sm})$ is the binary tetrahedral group.

$\pi_1(X^{sm})^{ab} \simeq \mathbb{Z}_3$.

Solution: Define $\mathrm{Cox}(X; x) = \bigoplus_{D \in \mathrm{Cl}(X_x^h)} H^0(X_x^h, D)$

Definition: The sing $(X; x)$ is a local MDS if $\mathrm{Cox}(X; x)$ is ess of finite type.

Thm (BCHM06): A Fano variety is a MDS.

Thm (BM21): A klt singularity is a local MDS.